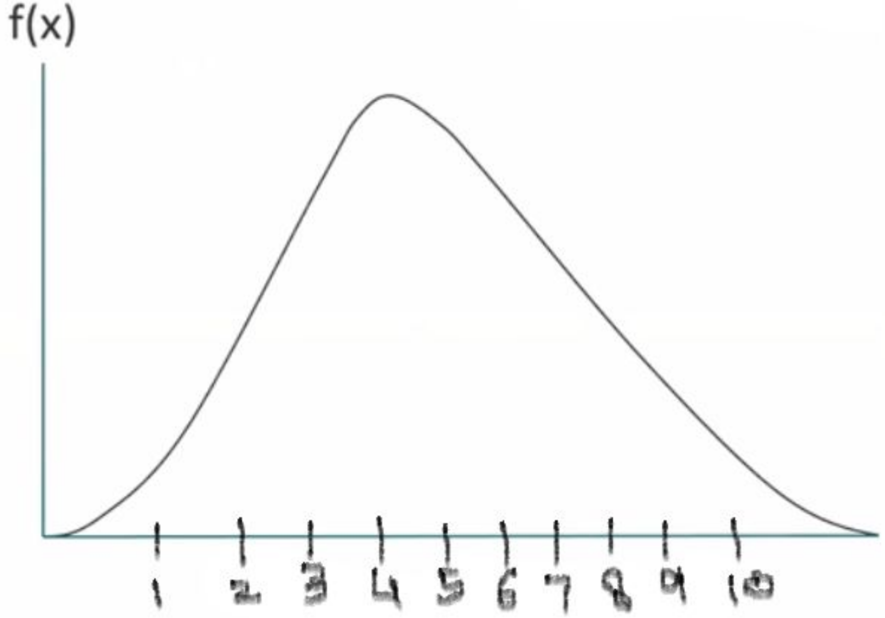
DESIGN LIFE

RELIABILITY

Design Life

- A design life is the time to failure t_R that corresponds to a specified reliability R

- What is the design life if a reliability of 0.9 is desired?



Problem

Given the following PDF for the random variable T, the time (in operating hours) to failure of a compressor, what is its reliability for a 100-hr operating life? What is the design life for 95% reliability?

$$f(t) = \begin{cases} \dfrac{0.001}{(0.001t + 1)^2} & t \geq 0 \\ 0 & otherwise \end{cases}$$

Solution

Problem

Given the following PDF for the random variable T, the time (in operating hours) to failure of a compressor, what is its reliability for a 100-hr operating life? What is the design life for 95% reliability?

$$f(t) = \begin{cases} \dfrac{0.001}{(0.001t + 1)^2} & t \geq 0 \\ 0 & otherwise \end{cases}$$

Solution

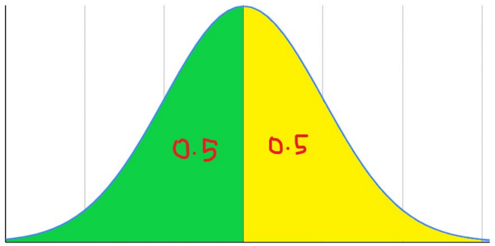
$$R(t) = \int_t^\infty f(t)dt$$

$$R(t) = \int_t^\infty \frac{0.001}{(0.001t + 1)^2}dt \qquad R(t) = \left.\frac{-1}{(0.001t + 1)}\right|_t^\infty \qquad R(t) = \frac{1}{(0.001t + 1)}$$

$$R(100) = \frac{1}{(0.001 \times 100 + 1)} = 0.909$$

$$R(t_R) = \frac{1}{(0.001t_R + 1)} = 0.95 \quad \Rightarrow \qquad t_R = 1000\{\frac{1}{R(t_R)} - 1\} = 52.6 \text{ hrs}$$

Median



- *Median* represents the middle value in the data where one-half the data is above and one-half the data is below when the data is ordered from lowest to highest

- The *median* of a continuous distribution, denoted by $\tilde{\mu}$ or t_{med} is the 50th percentile. So, $\tilde{\mu}$ satisfies

$$F(\tilde{\mu}) = R(\tilde{\mu}) = 0.5$$

That is, half the area under the density curve is to the $\frac{\text{Right}}{\text{Left}}$ of $\tilde{\mu}$

Example

- Consider the following pdf with x in hours

$$f(x)=\begin{cases} \frac{3}{8}(4x-2x^2), & 0<x<2, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the median.

$$F(x) = R(x) = 0.5$$

$$F(x) = \int_0^x \frac{3}{8}(4t - 2t^2)dt = 0.5$$

$$F(x) = \frac{3}{8}\left(2x^2 - \frac{2}{3}x^3\right) = 0.5 \quad \Rightarrow \quad x = 1$$

Mean Time To Failure (MTTF)

The expected or mean value of a continuous random variable T with pdf *f(t)* is

$$MTTF = E(T) = \int_0^{\infty} t f(t) dt$$

Mean Time To Failure (MTTF)

The expected or mean value of a continuous random variable T with pdf *f(t)* is

$$MTTF = E(T) = \int_0^\infty tf(t)dt$$

Since, $f(t) = -\frac{dR(t)}{dt}$ $\Rightarrow$ $MTTF = E(T) = \int_0^\infty -\frac{dR(t)}{dt}tdt = \int_0^\infty -tdR(t)$

Solving using integration by parts, $\int udv = uv - \int vdu$

$$MTTF = \int_0^\infty -tdR(t) = -tR(t)|_0^\infty + \int_0^\infty R(t)dt$$

$$MTTF = \int_0^\infty R(t)dt$$

Example

- Consider the following pdf with x in hours

$$f(x)=\begin{cases}\frac{3}{8}(4x-2x^2), & 0<\mathrm{x}<2,\\ 0, & \text{otherwise.}\end{cases}$$

- Find MTTF

$$MTTF=\int_0^\infty tf(t)dt \quad\Rightarrow\quad MTTF=\int_0^2 x\{\frac{3}{8}(4x-2x^2)\}dx$$

$$MTTF=\int_0^2 \frac{3}{8}(4x^2-2x^3)dx \quad\Rightarrow\quad MTTF=\frac{3}{8}\left(\frac{4x^3}{3}-\frac{2x^4}{4}\right)\Bigg|_0^2$$

$$MTTF=4-3=1$$

Problem

Consider the following pdf with t in hours

$$f(t) = \begin{cases} 0.002e^{-0.002t} & t \geq 0 \\ 0 & otherwise \end{cases}$$

Find R(t), MTTF and Median.

Problem

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$$f(t) = \begin{cases} 0.002e^{-0.002t} & t \geq 0 \\ 0 & otherwise \end{cases}$$

Find R(t), MTTF and Median.

Solution *R(t)* can be determine using

$$R(t) = \int_0^{\infty} f(t)dt \quad \Rightarrow \quad R(t) = \int_0^{\infty} 0.002e^{-0.002t}dt$$

$$R(t) = \left.\frac{0.002e^{-0.002t}}{-0.002}\right|_0^{\infty} \quad \Rightarrow \quad R(t) = e^{-0.002t}$$ Answer

$$MTTF = \int_0^{\infty} R(t)dt \quad \Rightarrow \quad MTTF = \left.\frac{e^{-0.002t}}{-0.002}\right|_0^{\infty} \quad \Rightarrow \quad MTTF = 500\ hrs$$ Answer

$$R(t_{med}) = e^{-0.002t} = 0.5 \quad \Rightarrow \quad t = 346.6\ hrs$$ Answer

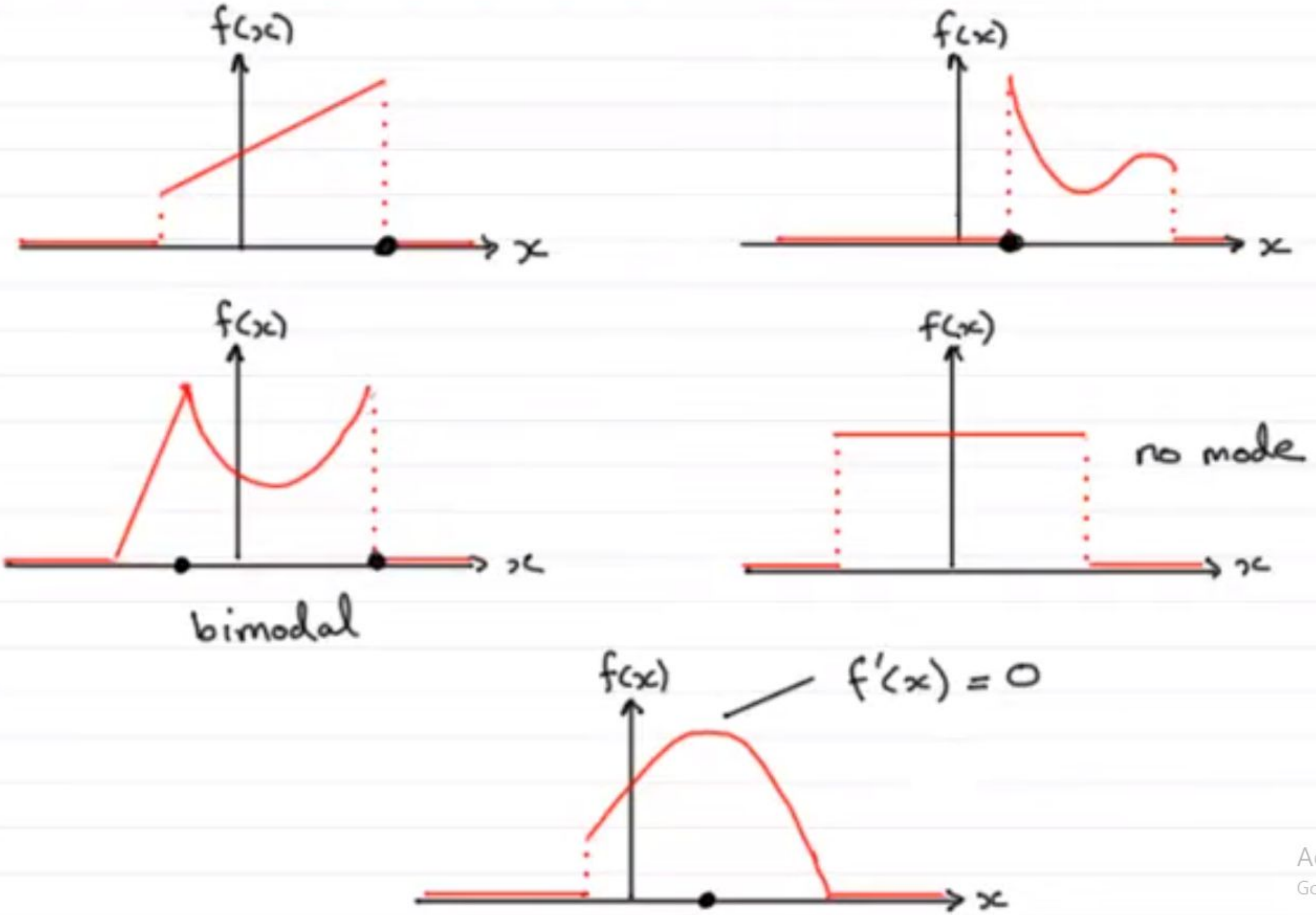
Mode

- Mode is the most likely observed failure time

$$f(t_{mode}) = \max_{0 \le t \le \infty} f(t)$$

- For a small fixed interval of time centered around mode, the probability of failure will be generally greater than that of an interval of the same size elsewhere within the distribution

The mode of a random variable *X* is the value of *x* below the highest point on a p.d.f



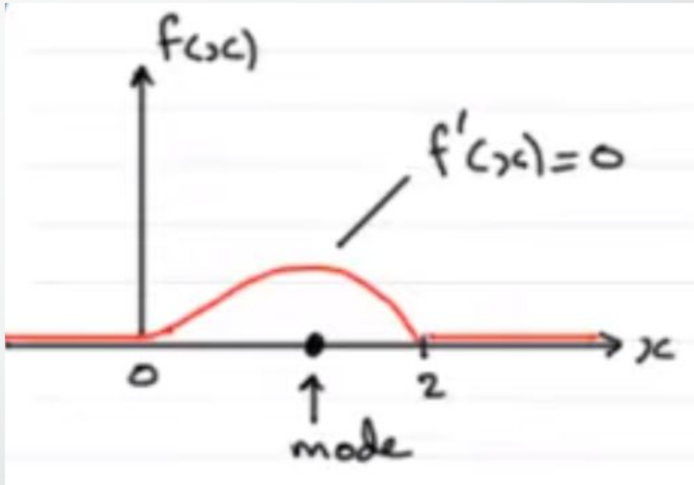
Problem

Find the Mode for given pdf

$$f(t) = \begin{cases} \frac{3}{4}x^2(2-x) & 0 \le x \le 2 \\ 0 & otherwise \end{cases}$$

Solution

To understand the response, pdf is first plotted



$$f(x) = \frac{3}{4}(2x^2 - x^3)$$

Differentiate the above function and putting equals to zero to determine Mode,

$$f'(x) = \frac{3}{4}(4x - 3x^2) \qquad Putting\ f'(x) = 0$$

$$\Rightarrow \qquad x = 0 \quad or \quad x = \frac{4}{3} \qquad \Rightarrow \qquad Mode = \frac{4}{3}$$

Answer

Problem

Find the Mode for given pdf

$$f(x) = \begin{cases} 6x(1-x) & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

Solution

Since pdf is given as follows:

$$f(x) = 6x(1-x)$$

Differentiate the above function and putting equals to zero to determine Mode,

$$f'(x) = 6(1-2x) \qquad Putting\ f'(x) = 0$$

$$\Rightarrow \qquad x = \frac{1}{2} \qquad \Rightarrow \qquad Mode = \frac{1}{2}$$

Answer

Problem

Consider the following pdf with t in hours

$$f(t) = \begin{cases} 0.002e^{-0.002t} & t \geq 0 \\ 0 & otherwise \end{cases}$$

Find Mode.

Solution *Since f(t)* is give as

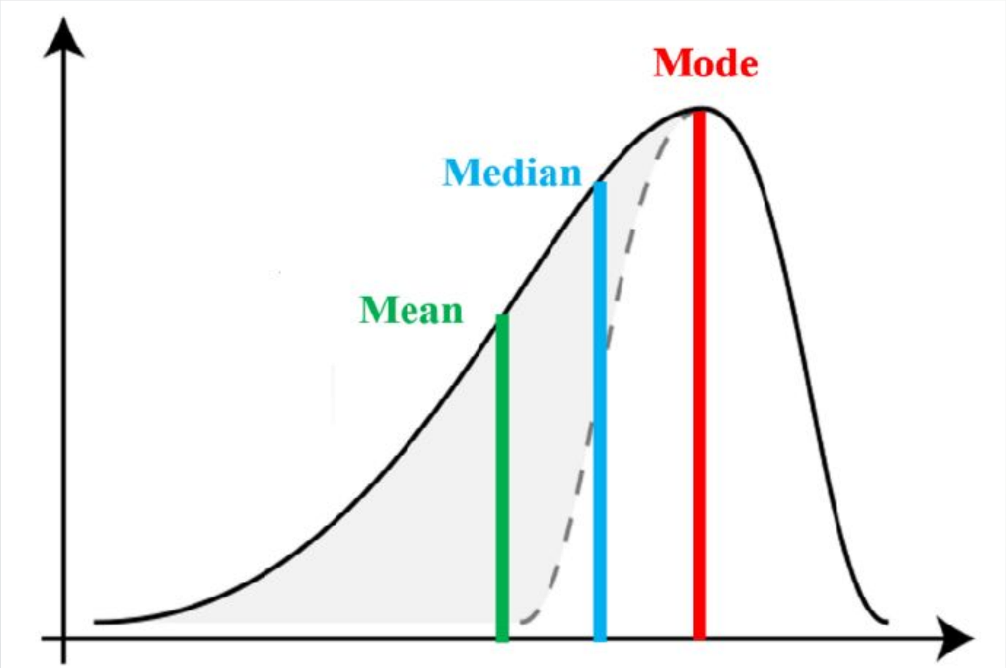
$$f(t) = 0.002e^{-0.002t}$$

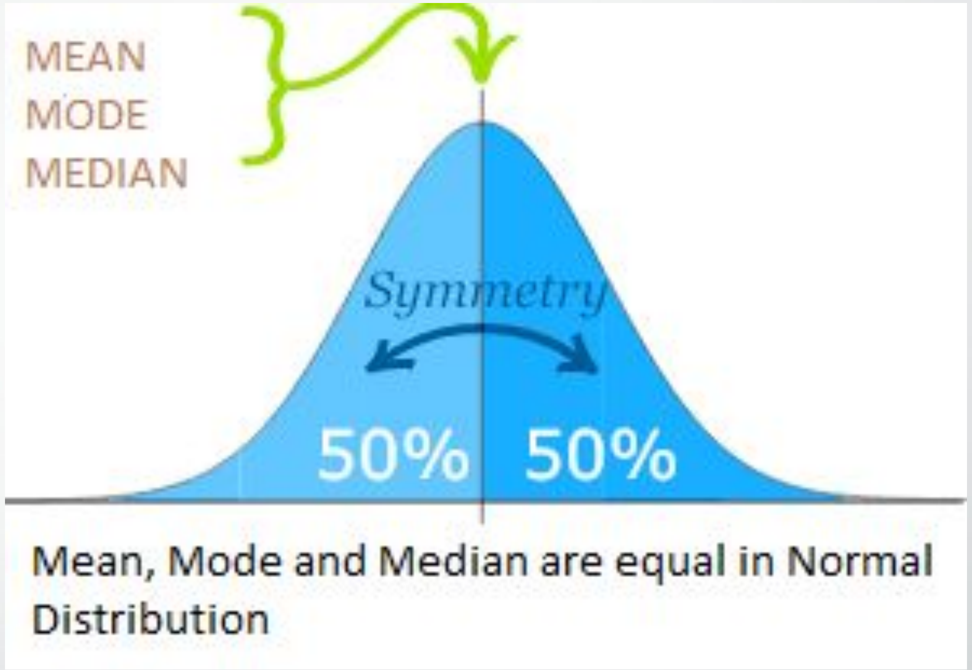
Differentiate the above function and putting equals to zero to determine Mode,

$$f'(x) = 0.002e^{-0.002t}(-0.002) \qquad Putting\ f'(x) = 0$$

$$\Rightarrow \qquad t = 0 \qquad \Rightarrow \qquad Mode = 0 \qquad \text{Answer}$$

COMPARISON OF MODE, MEDIAN AND MTTF





VARIANCE OR STANDARD DEVIATION

The expected variance of a continuous random variable t in terms of pdf is given by

$$\sigma^2 = \int_0^\infty (t - MTTF)^2 f(t)dt$$

$$\Rightarrow \qquad \sigma^2 = \int_0^\infty (t^2 - 2t\,MTTF + MTTF^2) f(t)dt$$

$$\Rightarrow \qquad \sigma^2 = \int_0^\infty t^2 f(t)dt - 2MTTF \int_0^\infty t\,f(t)dt + MTTF^2 \int_0^\infty f(t)dt$$

Since, $\int_0^\infty t\,f(t)dt = MTTF$ $\qquad \int_0^\infty f(t)dt = 1$

$$\sigma^2 = \int_0^\infty t^2 f(t)dt - MTTF^2$$

Problem

Consider the following pdf with t in hours

$$R(t) = \frac{1000 - t}{1000} \qquad 0 \leq t \leq 1000$$

Determine *f(t), MTTF and* variance.

Problem

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$$R(t) = \frac{1000 - t}{1000} \qquad 0 \leq t \leq 1000$$

Determine *f(t), MTTF and* variance.

Solution $MTTF = \int_0^{1000} R(t)dt \Rightarrow MTTF = 500\ hrs$

Since R(t) is give as $f(t) = -\frac{dR(t)}{dt} \Rightarrow f(t) = \frac{1}{1000}$

$$\sigma^2 = \int_0^{1000} t^2 f(t)dt - MTTF^2 \Rightarrow \sigma^2 = \int_0^{1000} t^2\left(\frac{1}{1000}\right)dt - 500^2$$

$$\sigma^2 = \frac{t^3}{3000}\Big|_0^{1000} - 500^2 \Rightarrow \sigma^2 = 83333.33$$ Answer

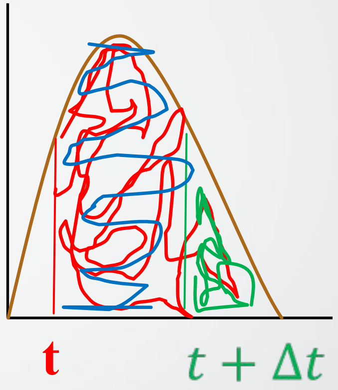
HAZARD FUNCTION

- Hazard function is a measure of the tendency to fail
  - The greater the hazard function, the greater the probability of approaching failure
- Hazard function is the instantaneous failure rate in very small time interval
  - From t_o to $t_o + \Delta t$ where $\Delta t \rightarrow 0$
- Hazard function is typically denoted by h(t) or $\lambda(t)$

HAZARD FUNCTION

- What is the probability that a system fails in $[t, t+\Delta t]$ in terms of the reliability function?

$$R(t) - R(t+\Delta t)$$



- What is the conditional probability of a failure in $[t, t+\Delta t]$ given the system survived up to t?

$$1 - R(\Delta t|t) = 1 - \frac{R(t+\Delta t)}{R(t)} = \frac{R(t) - R(t+\Delta t)}{R(t)}$$

- What is the conditional probability of a failure per unit of time i.e. failure rate?

$$\frac{R(t) - R(t+\Delta t)}{R(t)\,\Delta t}$$

HAZARD FUNCTION

- What happens if $\Delta t \to 0$?

$$\lim_{\Delta t \to 0} \frac{R(t) - R(t+\Delta t)}{R(t)\,\Delta t} = -\frac{1}{R(t)} \lim_{\Delta t \to 0} \frac{R(t+\Delta t) - R(t)}{\Delta t}$$

$$\Rightarrow -\frac{1}{R(t)} \frac{dR(t)}{dt} = \lambda(t) \qquad \Rightarrow \lambda(t) = \frac{f(t)}{R(t)}$$

$$Also, \qquad \lambda(t) = -\frac{d \ln R(t)}{dt} \qquad \Rightarrow \quad -\lambda(t)dt = d \ln R(t)$$

$$\Rightarrow \quad -\int_0^x \lambda(t)dt = \ln R(t)\Big|_0^x \qquad \Rightarrow \quad -\int_0^x \lambda(t)dt = \ln R(x) - \ln R(0)$$

$$R(x) = e^{-\int_0^x \lambda(t)dt}$$

Problem

Given the hazard function, where t is measured in operating hours. What is the design life if a 0.98 reliability is desired?

$$\lambda(t) = 5 \times 10^{-6} t$$

Solution

Since $$R(t) = e^{-\int_0^t \lambda(t)dt}$$

$$R(t) = e^{-\int_0^t 5\times10^{-6} t\, dt} \quad \Rightarrow \quad R(t) = e^{-\frac{5\times10^{-6}t^2}{2}}$$

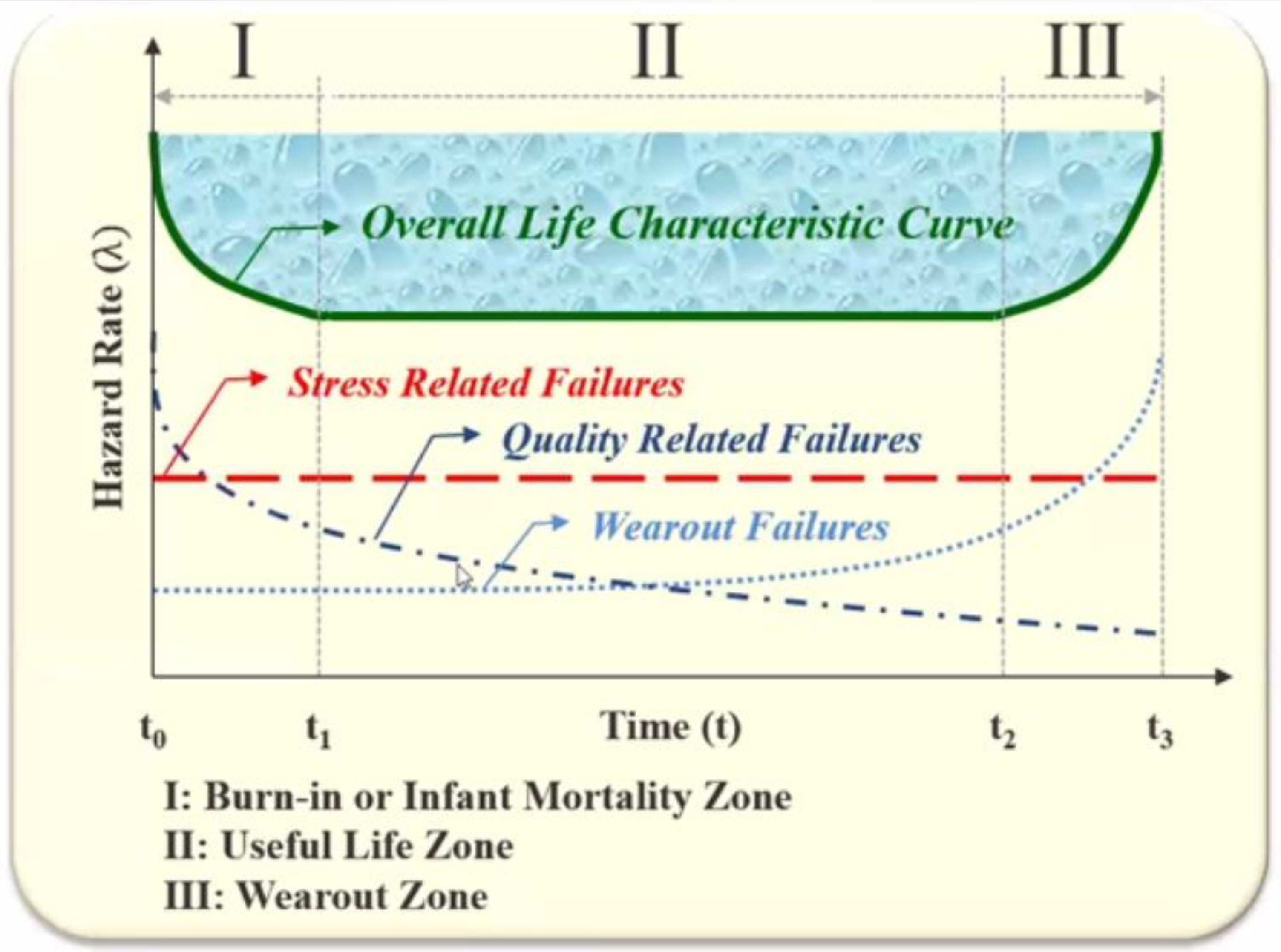
Since Design life is needed at 0.98 reliability i.e.,

$$\Rightarrow \quad 0.98 = e^{-\frac{5\times10^{-6}t^2}{2}} \quad \Rightarrow \quad \ln 0.98 = -\frac{5 \times 10^{-6} t^2}{2}$$

$$\Rightarrow \quad t_{0.98} = \sqrt{\frac{\ln 0.98}{-2.5 \times 10^{-6}}} \quad \Rightarrow \quad t_{0.98} = 89.9\ hours$$ Answer

HAZARD FUNCTION

Increasing Failure Rate (IFR)   Constant Failure Rate (CFR)
Decreasing Failure Rate (DFR)



	Hazard rate	Cause by	Reduced by
Burn-in	DFR	Manufacturing defects, Welding flaws, Cracks, Defective parts, Poor quality control, Contamination, Poor workmanship	Burn-in testing, Screening, Quality control, Acceptance testing
Useful life	CFR	Environment, Random loads, Human error, Chance events	Redundancy, Excess strength
Wearout	IFR	Fatigue, Corrosion, Aging, Friction, Cyclical loading	Derating, Preventive maintenance, Parts replacement

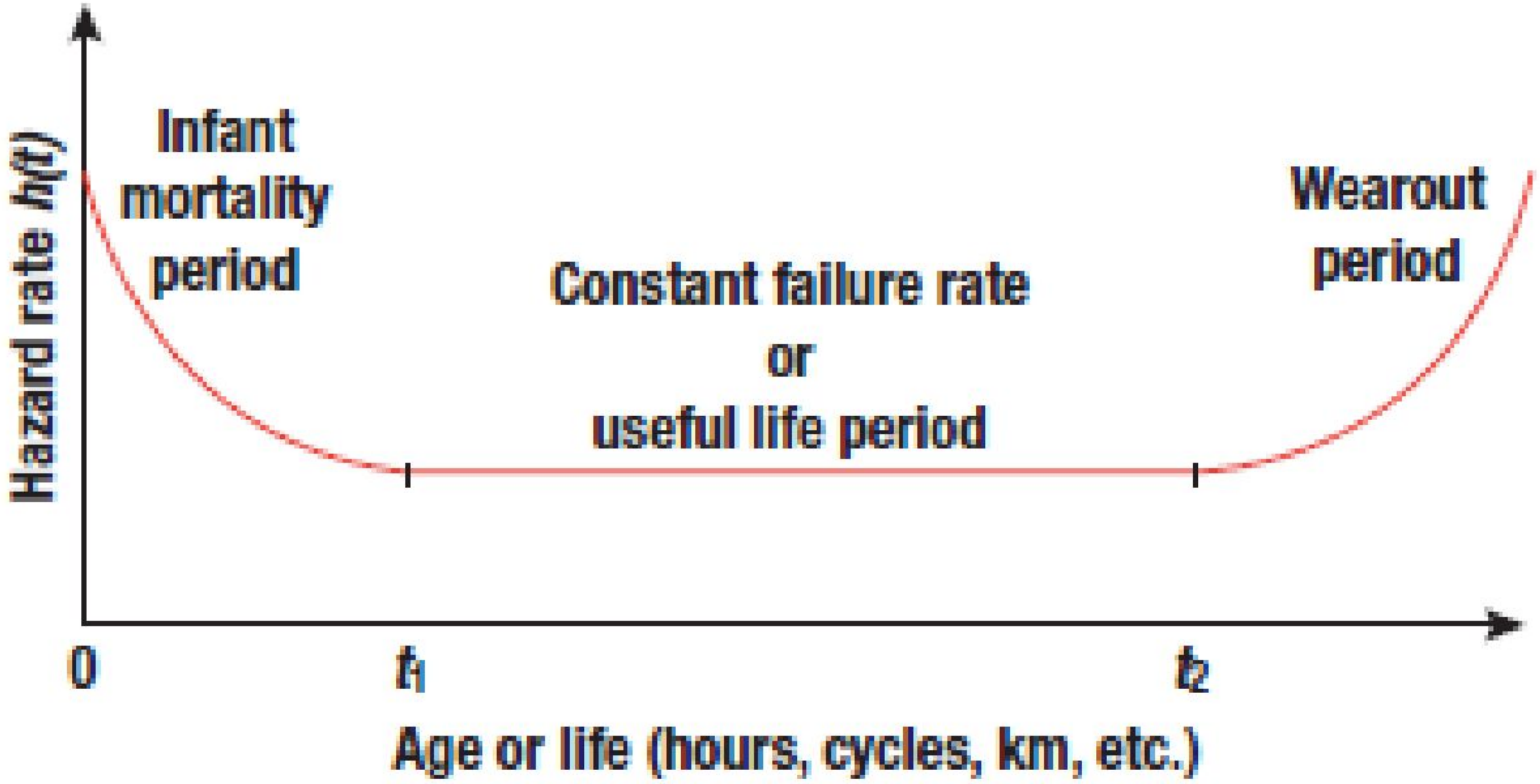
Relation between R(t), Q(t), F(t) , f(t) and h(t)

R(t) = Reliability function Q(t) = Unreliability function h(t) = hazard rate function

F(t) = failure Distribution function

f(t) = failure density function which indicates the rate of failures per hour

	R(t)	**Q(t) = F(t)**	**f(t)**	**h(t)**
R(t)	–	1- Q(t)	$-\int f(t)\,dt$	$e^{-\int_0^t h(t)\,dt}$
Q(t)	1- R(t)	–	$\int f(t)\,dt$	$1-e^{-\int_0^t h(t)\,dt}$
f(t)	$-\frac{d\,R(t)}{dt}$	$\frac{d\,Q(t)}{dt}$	–	$h(t)e^{-\int_0^t h(t)\,dt}$
h(t)	$-\frac{d}{dt}\ln R(t)$	$\frac{\frac{d\,Q(t)}{dt}}{1-\ Q(t)}$	$\frac{f(t)}{-\int f(t)dt}$	–


Hazard rate h(t)
Infant
mortality
period
Constant failure rate
or
useful life period
Wearout
period
0
t1
t2
Age or life (hours, cycles, km, etc.)

Hazard Rate

The failure of a population of fielded products can arise from inherent design weaknesses, manufacturing- and quality control-related problems, variability due to customer usage, the maintenance policies of the customer, and improper use or abuse of the product. The hazard rate, $h(t)$, is the number of failures per unit time per number of nonfailed products remaining at time t. An idealized (though rarely occurring) shape of the hazard rate of a product is the bathtub curve (Figure 2.6). A brief description of each of the three regions is given in the following:

1. *Infant Mortality Period.* The product population exhibits a hazard rate that decreases during this first period (sometimes called "burn-in," "infant mortality," or the "debugging period"). This hazard rate stabilizes at some value at time t_1 when the weak products in the population have failed. Some manufacturers provide a burn-in period for their products, as a means to eliminate a high proportion of initial or early failures.
2. *Useful Life Period.* The product population reaches its lowest hazard rate level and is characterized by an approximately constant hazard rate, which is often referred to as the "constant failure rate." This period is usually considered in the design phase.
3. *Wear-Out Period.* Time t_2 indicates the end of useful life and the start of the wear-out phase. After this point, the hazard rate increases. When the hazard rate becomes too high, replacement or repair of the population of products should be conducted. Replacement schedules are based on the recognition of this hazard rate.

Optimizing reliability must involve the consideration of the actual life-cycle periods. The actual hazard rate curve will be more complex in shape and may not even exhibit all of the three periods.

Generally, a bathtub curve can be divided into three regions. The *burn-in* early failure region exhibits a *decreasing failure rate* (DFR), characterized by early failures attributable to defects in design, manufacturing, or construction. Most components do not experience the early failure characteristic, so this part of the curve is representative of the population and not individual units. A time-to-failure distribution having a DFR is referred to as a distribution belonging to the class of DFR distribution.

Analogously, a time-to-failure distribution having a decreasing average failure rate is referred to as a distribution belonging to the class of decreasing failure rate average (DFRA) distribution.

The *chance-failure region* of the bathtub curve exhibits a reasonably *constant failure rate*, characterized by random failures of the component. In this period, many mechanisms of failure due to complex underlying physical, chemical, or nuclear phenomena give rise to this approximately constant failure rate. The third region, called the *wear out region*, which exhibits an *increasing failure rate* (IFR), is characterized mainly by complex aging phenomena. Here, the component deteriorates (e.g., due to accumulated fatigue) and is more vulnerable to outside shocks. It is helpful to note that these three regions can be radically different for different types of components. Figures 3.2 and 3.3 show typical bathtub curves for mechanical and electrical devices, respectively.

These figures demonstrate that electrical devices can exhibit a relatively large chance-failure period. Figure 3.4 shows the effect of various levels of stress on a device.

As the stress level increases, the chance-failure region decreases and premature wear out occurs. Therefore, it is important to minimize stress factors, such as a harsh operating environment, to maximize reliability.

Similar to the DFR and DFRA distributions, the IFR and *increasing failure rate average* (IFRA) distributions are considered in the framework of a mathematical theory of reliability [1].

Table 3.1 lists the cdfs (unreliability functions) and hazard rate functions for important pdfs.

FIGURE 3.2 Typical bathtub curve for electrical devices.

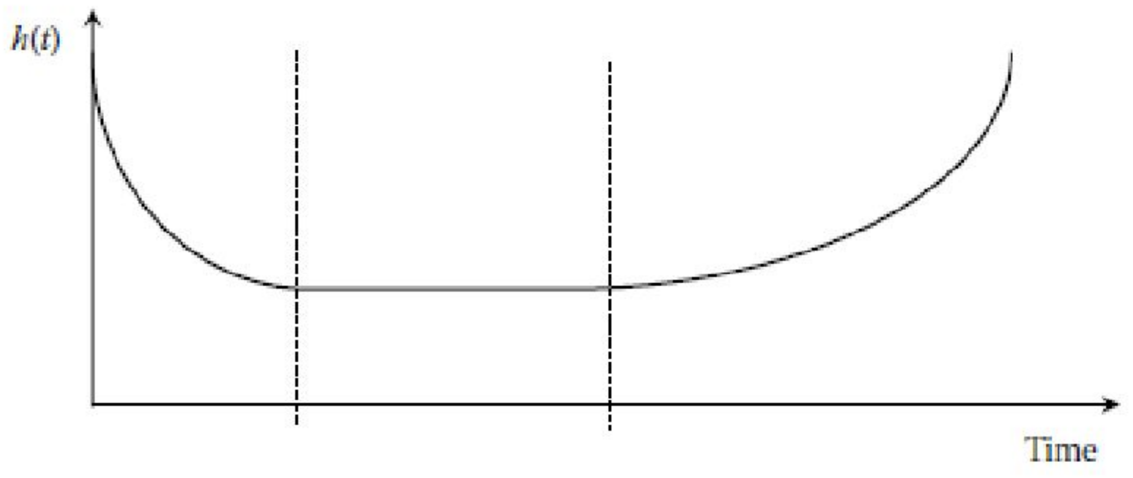
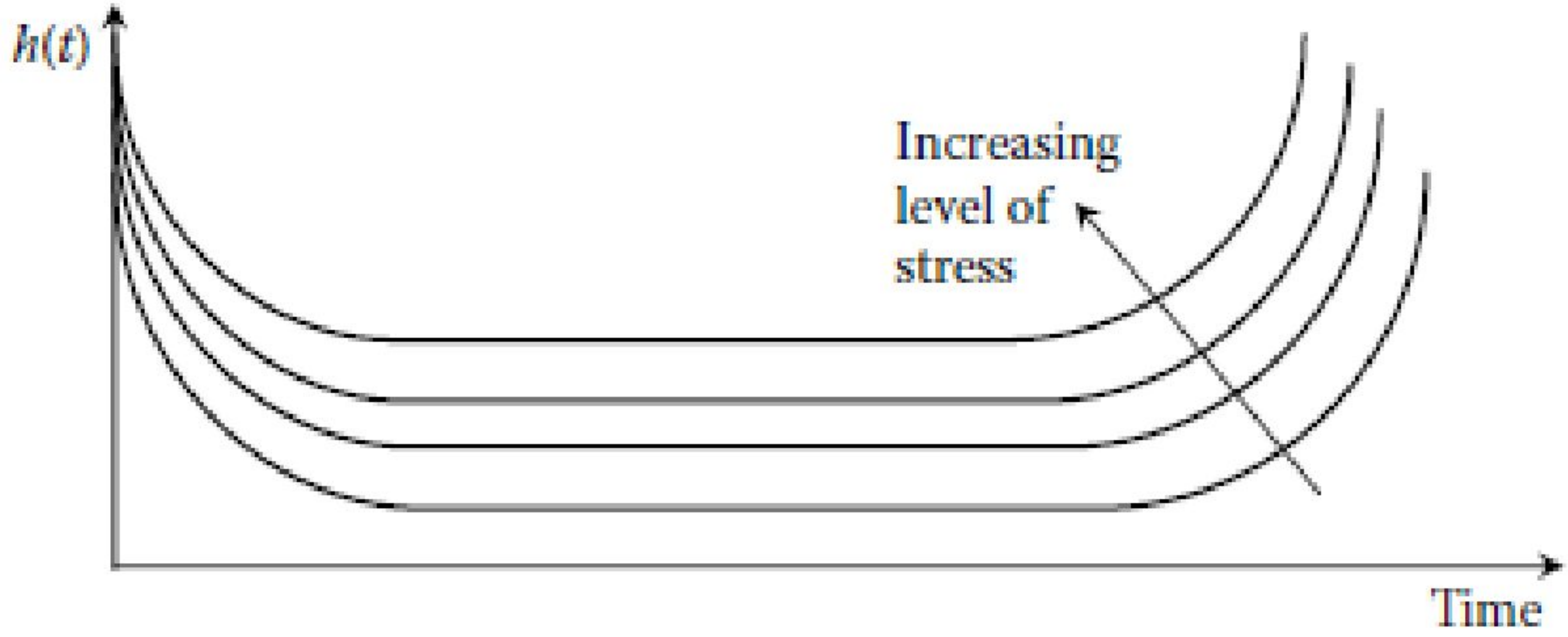


FIGURE 3.3 Typical bathtub curve for mechanical devices.



Effect of stress in a typical bathtub curve.

Probability Distributions to Model Failure Rate

Exponential Distribution The life-cycle curve of Figure 11-1 shows the variation of the failure rate as a function of time. For the chance-failure phase, which represents the useful life of the product, the failure rate is constant. As a result, the **exponential distribution** can be used to describe the time to failure of the product for this phase. In Chapter 4 the exponential distribution was shown to have a probability density function given by

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0 \tag{11-1}$$

where λ denotes the failure rate. Figure 4-21 shows this density function.

The **mean time to failure** (MTTF) for the exponential distribution is given as

$$\text{MTTF} = \frac{1}{\lambda} \tag{11-2}$$

Thus, if the failure rate is constant, the mean time to failure is the reciprocal of the failure rate. For repairable equipment, this is also equal to the **mean time between failures** (MTBF). There will be a difference between MTBF and MTTF only if there is a significant repair or replacement time upon failure of the product.

The reliability at time t, $R(t)$, is the probability of the product lasting up to at least time t. It is given by

$$R(t) = 1 - F(t) = 1 - \int_0^t e^{-\lambda t} dt = e^{-\lambda t} \tag{11-3}$$

where $F(t)$ represents the cumulative distribution function at time t. Figure 11-2 shows the reliability function, $R(t)$, for the exponential failure distribution. At time 0, the reliability is 1, as it should be. Reliability decreases exponentially with time.

In general, the **failure-rate function** $r(t)$ is given by the ratio of the time-to-failure probability density function to the reliability function. We have

$$r(t) = \frac{f(t)}{R(t)} \tag{11-4}$$

For the exponential failure distribution

$$r(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

implying a **constant failure rate**, as mentioned earlier.

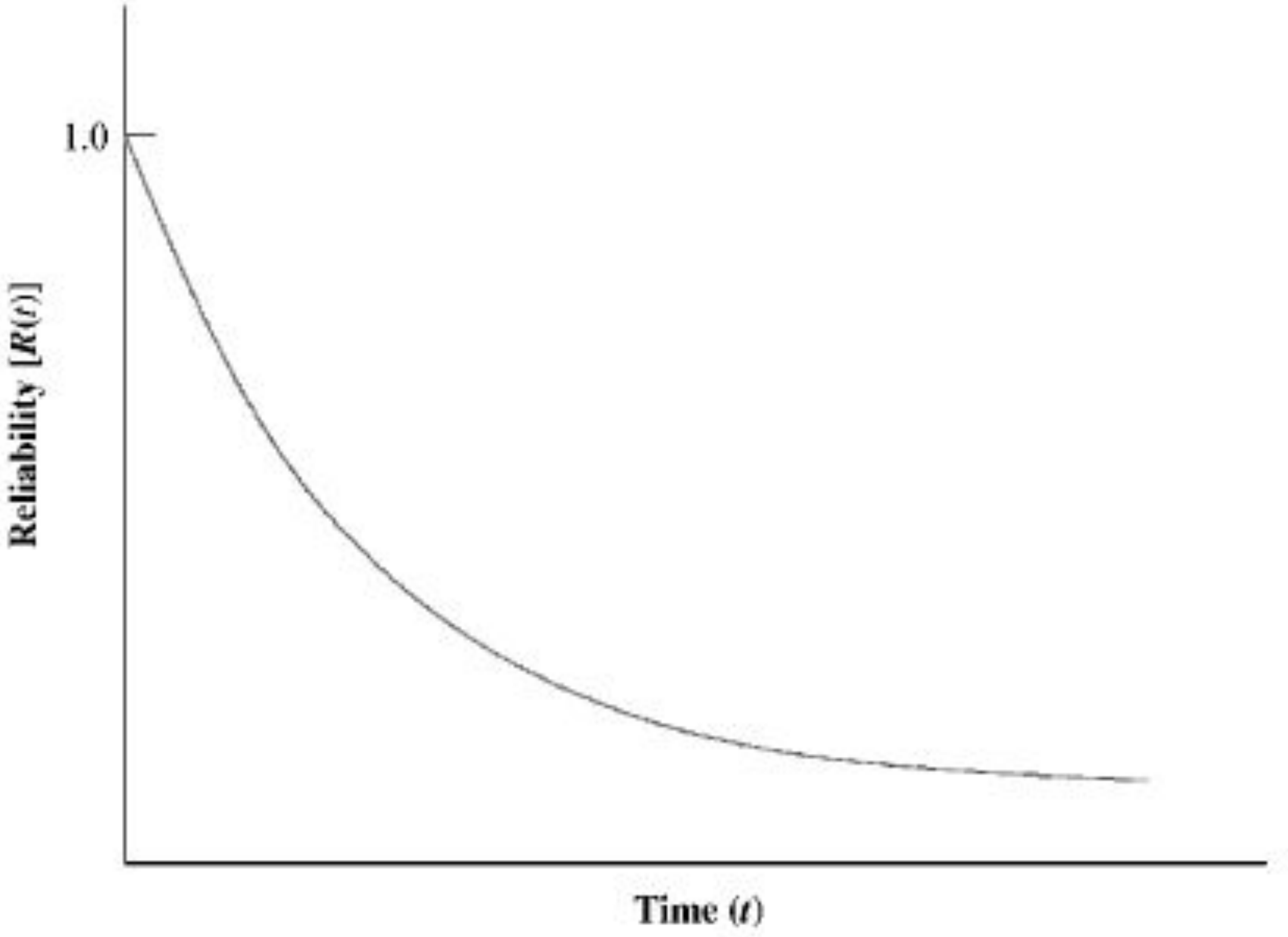
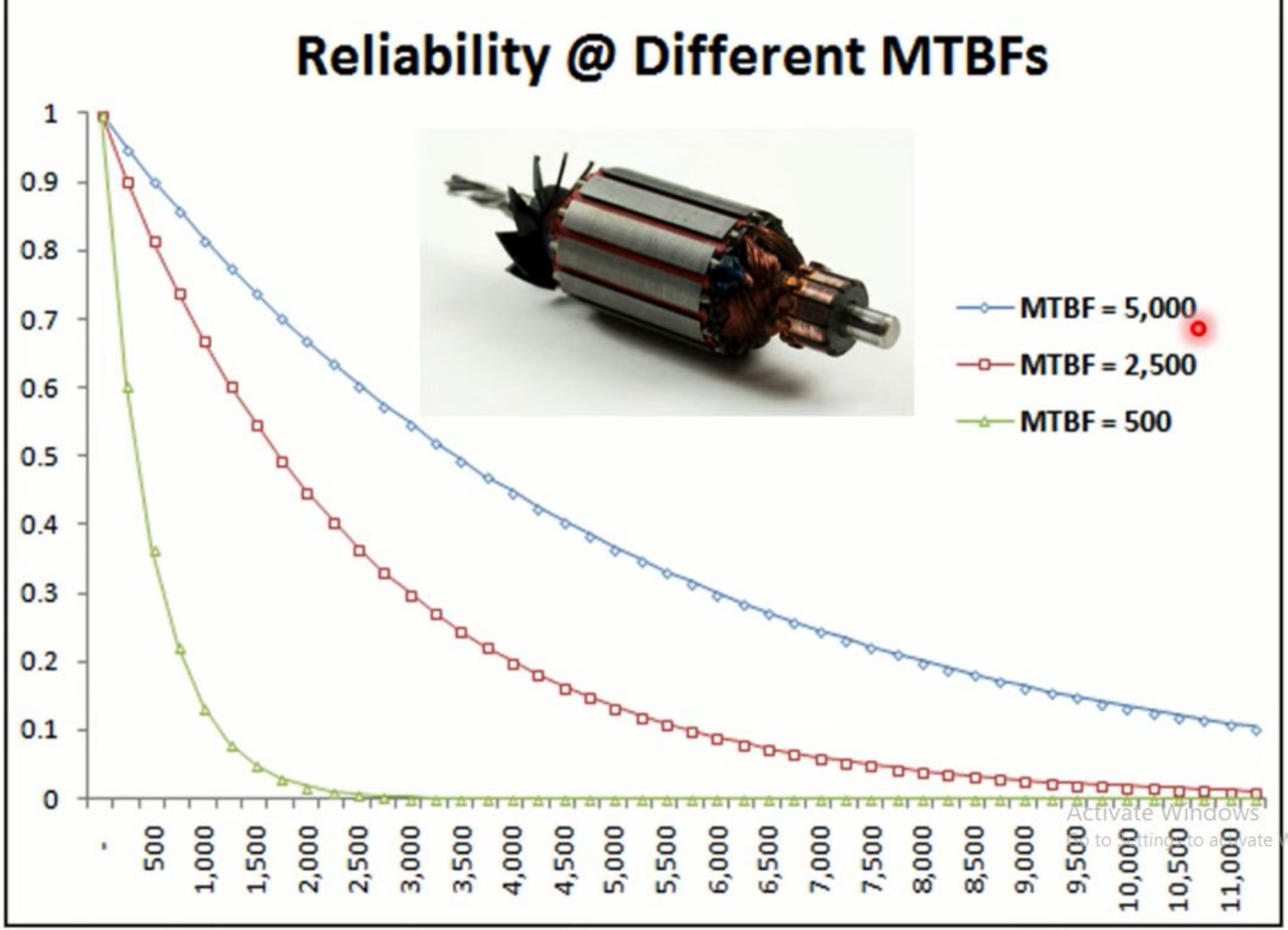


FIGURE 11-2 Reliability function for the exponential time-to-failure distribution.



Reliability @ Different MTBFs
MTBF = 5,000
MTBF = 2,500
MTBF = 500
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-
500
1,000
1,500
2,000
2,500
3,000
3,500
4,000
4,500
5,000
5,500
6,000
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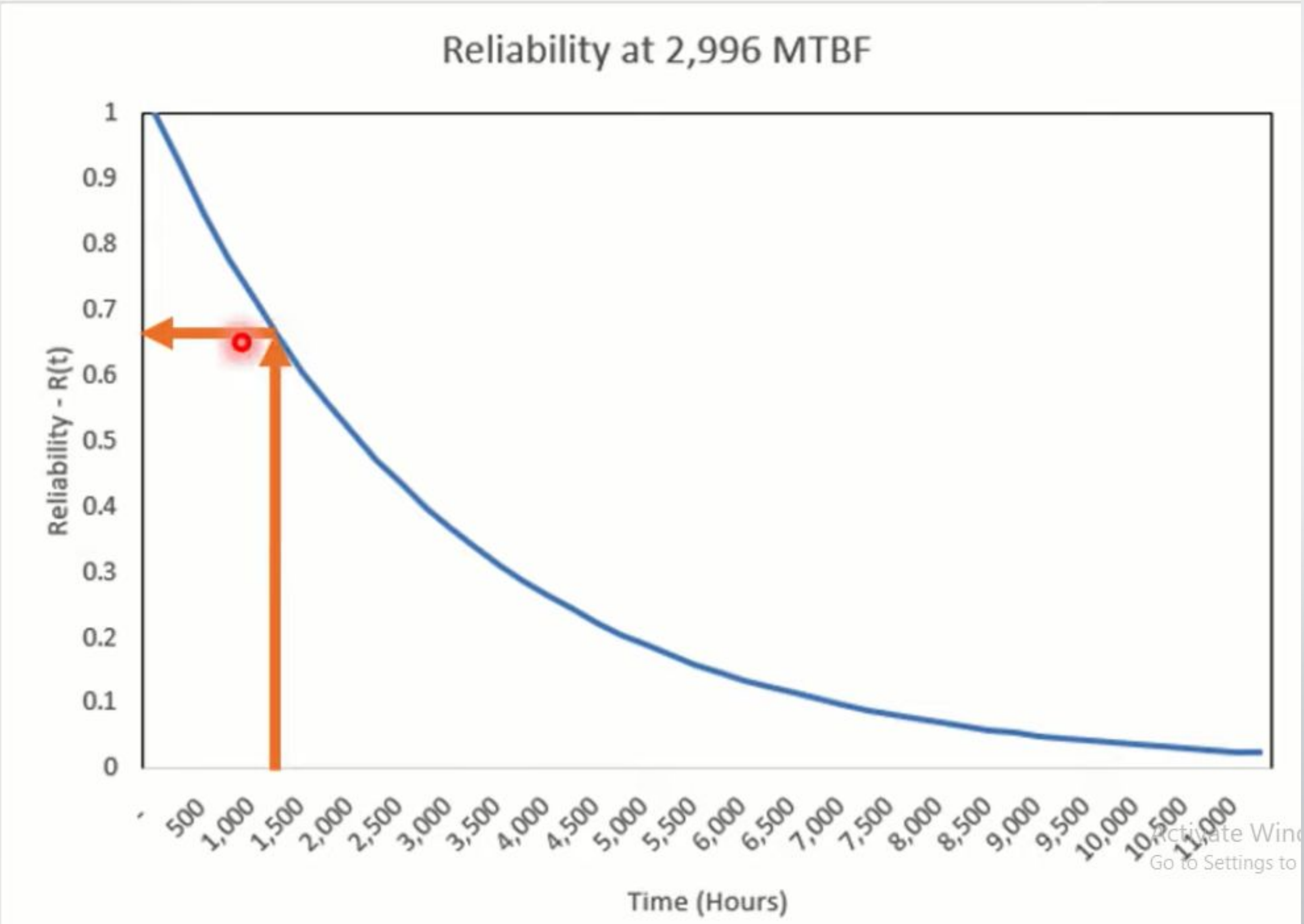
9,000
9,500
10,000
10,500
11,000


Reliability at 2,996 MTBF
Reliability - R(t)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-
500
1,000
1,500
2,000
2,500
3,000
3,500
4,000
4,500
5,000
5,500
6,000
6,500
7,000
7,500
8,000
8,500
9,000
9,500
10,000
10,500
11,000
Time (Hours)